How the representations take on a key role in an inclusive educational sequence concerning fraction.

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The design analysis of an inclusive educational sequence concerning the teaching and learning of fractions (Robotti, et al., 2015) is the focus of this work. Referring to the principles of Universal Design for Learning, developed by the Centre for Applied Special Technology to reduce barriers in learning, we analyse the design of the educational sequence focussing on fractions and devoted to classes where students with certifications of mathematical learning disabilities (MLD) were present. From the point of view of mathematics education, we refer to the theory of semiotic mediation (Bartolini-Bussi, Mariotti, 1998) in order to clarify the cognitive role of artefacts taken into account in class activities. In particular, the use of artefacts to solve tasks produces representations that contribute to develop of mathematical meanings aimed in the teaching activity.

Keywords: Mathematical learning disabilities, Universal Design for Learning, fraction, number line, artefact.

Introduction and conceptual framework

The focus of this paper is the analysis of an inclusive educational sequence concerning the teaching and learning of fractions already described in detail in CERME9 (Robotti, *et al.,* 2015). In the paper presented in CERME9 we discussed how, in the context of Semiotic Mediation (Bartolini, Mariotti, 2008), the choice of particular artifacts and the design of tasks related to their use, allowed students to grasp different meanings of fraction. Now, I would like to discuss why that educational sequence was "inclusive" for learners with certifications of mathematical learning disabilities (MLD) or difficulties in math or with low achievement in math. To this aim, I will refer to the theoretical framework of Universal Design for Learning (UDL) and I will analyse how the design of the activities follows the guidelines and principles of UDL in order to reduce barriers in learning. Therefore, the rationale for referring both theoretical frameworks is to consider learning difficulties in a context of math education: if the Semiotic Mediation framework allowed us to design a structured educational sequence for teaching and learning different of meanings of fraction (for this, see CERME9 proceedings), the UDL framework, allowed us to design and analyse those tasks in order to be inclusive for MLD students, students with difficulties or with low achievements. Even if there isn't consensus on definition and identification of MLD students (Karagiannakis, *et al.,* 2014) and the inclusivity (Ianes, 2006) is not a construct used consistently across different fields (education, society...) or in different countries, in this research work, we considered as "inclusive educational activities" the educational activities, developed in the context of the class, which face to the special needs of MLD students including dyscalculic students, students with difficulties in math and students with low achievement in math. In other words, according to the UDL framework, we consider "inclusive educational activities" those that meet the needs of all students of the class. UDL is based on a set of principles and guidelines that have been elaborated to increase access to curriculum for all the students, including those with disabilities. These principles and guidelines have a general valence and they are devoted to various application contexts. In this paper we refer to the context of math education. At the core of UDL is the belief that there are three networks in our brain that support learning: (i) The Knowledge/Recognition Networks, which are involved in identifying and interpreting sound, light, taste, smell, and touch. They are essential to learning because students are expected to comprehend a text, interpret formulas, identify cause/effect relationships, etc. UDL states that it is possible to support the knowledge/recognition networks providing Multiple Means of Representation (1° principle) to give learners various ways of acquiring information and knowledge. This principle has three supporting guidelines: provide options for Perception; provide options for Language, Mathematical expression, and symbols; provide options for Comprehension; (ii) The Strategic Networks, which are involved in planning, executing, and monitoring actions and skills. In learning, they occur, for example, to solve a problem, use an artefact, take notes and listen to a lecture. UDL states that it is possible to support the Strategic networks providing Multiple Means of Action and Expression (2° principle) to give learners alternatives for demonstrating what they know. This principle has three supporting guidelines: provide options for Physical action; provide options for Expressive skills and fluency; provide options for Executive function; (iii) The Affective Networks, which are responsible for establishing priorities and interests. UDL states that it is possible to support the Affective networks providing Multiple Means of Engagements (3° principle) to stimulate interest and motivation for learning. This principle has three supporting guidelines: provide options for Physical action Recruiting interest; provide options for Physical action Sustaining effort and persistence; provide options for Physical action Self-regulation. According to UDL frameworks, representations seem to be "the ways [to] perceive and comprehend information", produced by visual or auditory means, or contained in a printed text. Therefore, UDL considers representations as signs of different nature (visual, auditory, kinaesthetic…) produced by different means (or artefacts). In the Theory of Semiotic Mediation, Bartolini and Martiotti state that: "[…] any representation comes to life because of a human construction that makes it possible, in other words any representation is supported by an artefact" (Bartolini, Mariotti, 2008, p. 747). Even if the difficulty in articulating an accurate definition for the term "representation" is recognized in math education, we want to stress that, in his theoretical framework Duval states the importance of connections both within and amongst different representational registers as absolutely fundamental to deep understanding of mathematics (Duval, 1999). According to this, Arcavi (2003) stresses the essential role of visual representations in the learning of mathematics and he defines "visualization, as both the product and the process of creation, interpretation and reflection upon pictures and images, …" (Arcavi, 2003, p. 215). Visualization, visual representations and more general representations were taken into account in our analysis presented in CERME 9 discussion. As described in that analysis, the teaching of the notion of fraction is a quite delicate issue, which requires insightful ways of structuring didactical activities. In that occasion, we stressed how the importance of spatial processes, performed on the base of spatial skills, can be important in mathematical performances where explicit or implicit visualization is required, as in the case of learning of fractions. Therefore, the visual non-verbal, the kinaesthetic-tactile and the auditory channels were considered as preferential for designing inclusive educational activities concerning fractions. Finally, Goldin (1992) outlined a unified model for the psychology of mathematics learning, which incorporated cognitive and affective attributes of visualization as essential components in systems of representation in mathematical problem solving processes. From the above considerations, we can observe that, even in the domain of math education, the aspects related to multiple means of representations and their relations, multiple means of action and expression and multiple means of engagement are taken into account. For this reason, we use the three principles of UDL to analyse the design of an educational sequence about fractions whose purpose was to be inclusive (in the UDL sense), that is to say, to meet the needs of all learners of the classes, even MLD students. Moreover, we will adopt the framework of Semiotic Mediation to analyse the role of the artefacts in the educational sequence in order to reach the aimed educational goals. More in detail, we will show how the design of the different tasks, requiring actions on artefacts, can be interpreted through the three principles of the UDL to analyse the efficacy both of construction of meanings for the notion of fraction and of inclusion for MLD students.

An example of inclusive educational sequence

I will briefly recall the main activities characterising the educational sequence concerning fractions, described in details in TWG13 of CERME 9 (Robotti & al., 2015), which was carried out in 22 classes (nine 5th grade classes, six 4th grade classes, and seven 3rd grade classes) involving around 400 students, of which 20 were certified MLD students (Robotti & al., 2016): (1) Partitioning of the A4 sheet of paper (named "placemat"). (2) Partitioning of a strip of squared paper: given a certain unit of measure (number of squares) on the strip, position a fraction on that strip (1/2 or 1/3 or 1/4, ….); given different units of measure on different strips, on each strip a same fraction is represented (1/2); chosen an appropriate unit of measure on the strip, different fractions are represented on that strip (e.g., 1/3 and 1/5). (3) Placing fractions on the number line. (4) Placing coloured tags, labelled with fractions, on a "string on the wall".

The learning difficulties concerning fractions seem to be due, among other things, to the lack of connection between different meanings of fraction (Charalambous, Pitta-Pantazi, 2005) and to the fact that only some of them are fostered (e.g. the meaning of "part/whole"), (Pantziara, Philippou, 2011). Moreover, some visual representations are favoured (such that of the pizza) even if they could hinder the learning of the different meanings of fraction (Fandiño Pinilla, 2007). The main learning difficulties identified in Italian national assessment (INVALSI tests) are related to: managing the meaning of the "equal" sign (for instance, What does it means obtaining "equal parts" of the whole?); switching from a fraction to the unit that has generated it; managing equivalent fractions; ordering fractions on a straight-line even without transform them to decimal numbers. In the following, I will discuss how principles and guidelines of UDL can be effectively used to analyse the design of the tasks, which allow overcoming these difficulties. The analysis of actions required by tasks on the artefacts, will show how the educational aims and the aims of inclusion were achieved.

The artefact "placemat"

The "placemat" is a A4 sheet of paper that, at the beginning of the sequence will be in white colour but that, in the following, will be coloured. The mathematical meanings to be mediated by the “placemat” are: 1) construction of fractional units starting from a given unit of measure (the A4 sheet of paper) by folding and cutting out the A4 sheet of paper; 2) equivalence between fractional units, by folding or cutting the fractional units in order to show the equivalence between the surfaces. This allows to overcome the difficulty related to the interpretation of “equal parts of the whole” as “congruent fractional units” instead of “equivalent fractional units” (see above); 3) sum of fractional units in order to obtain the given unit of measure (in this case, the A4 sheet of paper). This can be realized covering the A4 sheet with different fractional units. This allows to come back to the unit of measure starting from the fractional units. According to Guideline 1 of first Principle of UDL, learning is difficult if information is imperceptible to the learner, or when information is presented in formats that require extra effort for him/her (for example, decoding the text “one half” or decoding arithmetical expressions “1/2” for a dyslexic student). To reduce barriers to learning, it is important to ensure that key information are equally perceptible to all learners. For this reason we provided the same information through different modalities (e.g., through the possibility of touching and manipulating the pieces of A4 sheet, through vision of their drawing on the notebook, through hearing or reading an arithmetical expression). In other words, through different forms of representation (Guideline 2, Principle 1). Moreover, the colour, which characterises fractional units later on, plays the role of support to long-term memory for students with learning disabilities or simply students with math impairment. Once the information are made accessible, our educational sequence aims to help students transform them into useable knowledge. As pointed out by the Guideline 3 of the first Principle of UDL, this depends upon “information processing skills” like: selecting useful information, integrating new information with prior knowledge, strategic categorization and active memorization. Individuals differ greatly in their skills in information processing, but effective design of task and presentation of information in accessible ways can provide the scaffolds necessary to ensure that all learners are able to process information. Therefore, we designed educational activities in which students have to act on the artefact, in order to produce representations (for instance, by cutting or folding different pieces of paper referring to the same fractional unit in order to show the equivalence of their surfaces or choosing appropriate fractional units to cover the A4 sheet in order to show that the sum of appropriate fractional units gives the unit of measure 1), and they have to put the obtained representations into relation. Indeed, the actions, performed by students on the artefacts, produce situated signs (representations) through which students, with the help of teacher’s mediation, construct the mathematical meanings aimed for (in that case, equivalent fractions or 2/8+3/4=1). In order to help students remember the mathematical meanings described, the teacher asks students to: reproduce on the note-book the operations performed with the artefact; write down the content of the class discussions; write texts in which the processes developed in the activity are explicitly linked to knowledge. This allows recalling and using knowledge in the future and it provides options for expression and communication. As a matter of fact, it is well known that there is no medium of expression that is equally suited for all learners and for all kinds of communication. This means that, according to the second Principle of UDL, students with learning disabilities may excel, for example, in interpretation of drawing data (e.g., the drawing of “placemat” covered by different coloured fractional units), but they may falter when asked to read data provided in a table or in arithmetical expression (e.g. 1= 2/8+3/4).

The artefact "strip of squared paper"

The strip of squared paper is a strip with squares of 1 cm, 10 cm high and approximately 1 m long. In the strip there are some integer numbers (0, 1, 2, 3, …) and fractions are constructed. The mathematical meanings mediated by the "strip of squared paper" are: 1) Fraction as operator on a given unit of measure: once a unit of measure is considered, the teacher asks students to place on the strip a given fractional unit and then a given fraction (equal o more than 1); Comparison, by means of a perceptive strategy, of fractional units that should be positioned on different strips of paper; 3) Relationship between a fractional unit and a chosen unit of measure: considering different units of measure on different strips, the teacher asks to place the same fractional unit (for instance, ½) on all the strips; 4) Ordering fractional units: considering on a strip an appropriate unit of measure (given by the l.m.d. of the denominators), the teacher asks to place different fractional units. This allows students to compare them in a concrete way; 5) Equivalence between fractions: considering on a strip a given unit of measure, the teacher asks to place different fractions. Among them there are some equivalent fractions. We note that some mathematical meanings concerning fraction, such as equivalent fractions or comparison of fractions, are already presented in previous activities with the artefact "placemat". Once again, the first principle of the UDL is used in this educational sequence providing different means of representation (that is, different artefacts through which representations can be performed). In this activity, the students produce different representations: linguistic signs associated to the name of the fraction expressed in verbal language ("One half"), in verbal visual language (the writing "One half") and arithmetical language ("1/2"). The teacher institutionalizes the relationship between the different signs in terms of rational numbers. Thus, the construction of meaning related to the notion of rational number, is based on both the interplay between different types of semiotic representations, according to the first principle of UDL, and by different actions on artefacts producing those representations, according to the second principle of UDL. Moreover, we note that the colour, perceptive option already provided in the A4 sheet of paper, is used with the same aim also in the activities with strip of squared paper. Thus, the fractional units constructed on the strip are coloured with the same colour of the respective fractional units constructed by A4 sheet of paper. Therefore, the colour assumes the role of a support for memory and also of an artefact, which allows students to link the meaning of fractional unit constructed with the "placemat" (fraction as a *part of a whole*) with the meaning of fractional unit constructed by the strip of paper (fraction as *operator*). Moreover, we have taken into account different strips where different units of measure have been considered and where have been represented a given fractional units. The strips are put one beside the other (proving option for physical actions, as called for by the second principle of UDL) and the relationship between unit of measure and fractional units becomes perceptively evident, reifying Guideline 1 (provide options for perception) of the first principle of the UDL. In order to link the meaning of fraction as part of a whole and as an operator to the meaning of rational number on a number line, we need to switch from the artefact "strip of squared paper" to the artefact "number line". For this reason, the teacher asks the students to represent, on the *same* strip different fractional units: 1/8, 1/6, ¼, 1/3 and ½. Thus,, in order to represent all fractional units on the same strip, students need to find a suitable graphic strategy that maintains the colour without superposing different colours on the strip. They adopt coloured notches. Positioning on single strip different fractions makes the ordering of fractions exactly like that of the other perceptively evident numbers. Here the semiotic potential of the artefact "strip of squared paper" associated to the tasks proposed by the teacher becomes evident, playing a key role in fostering identification of fractions as rational number on the number line. I will show this in the following session.

The artefact "number line"

The number line is drawn by students on their note-book. It is a number line of positive numbers starting from the point 0 and it is presented as the natural "crushing" of the paper strip. Therefore, on the number line, students place fractions without transforming them into decimal numbers. The mathematical meanings supported by the "number line" and the tasks proposed are similar to those proposed in activities with the artefact "paper strip". This allows all students to find continuity both between the artifacts proposed and the construction of the aimed for mathematical meanings.

The artefact "string" on the wall

The artefact "string on the wall" consists of a string in nylon whose ends are fixed to two adjacent walls of the class (such as a wire for drying clothes). On it, students hang some tags labelled with fractions (in this case, tags are coloured accordingly with the colour used in previous activities) or natural numbers (in this case tags are white). The string simulates the number line starting from zero, which is placed at the left end and the position of the unit is made to vary dynamically sliding the corresponding labelled tag attached with a clothes' peg. In this case, however, the positions of the other tags do not vary dynamically at the same time or automatically as a consequence of the new placement of the unit: their motion requires a specific action in order for the numbers on the line to maintain the desired mathematical relationships. Tags are made so that they can be hung in "clusters" to ensure that tags corresponding to equivalent fractions have the same position on the string. The mathematical meanings supported by the "string" are: 1) Ordering of fractional units or fractions: once the unit of measure has been defined, by positioning the tag labelled with "1", the tags labelled with fractions need to be hung in the correct position; 2) Equivalent fractions: since their tags correspond to the same position on the string, they are hanging as "clusters" and they represent classes of equivalence. Students named them "caterpillars" whose first tag (corresponding to the irreducible fraction) was called "head of the class"; 3) Density of the numerical set Q: "enlarging" the unit of measure, that is to say increasing the distance between the tag corresponding to 0 and that corresponding to 1, it is possible to hang on the strip more and more fractions. This operation, repeated many times, allows constructing a mental image connected to the idea of infinity. We note that, when tags on the strip become numerous, one needs to "enlarge" their position to make room for other tags. This action on the string is a situated sign used by the teacher to introduce the idea of density in the set Q. Therefore, the representations of fraction on the string (tags labelled with fractions), their position on the string and the action performed on the tags are in line with the first principle of the UDL according to which different forms of representation are needed in order to capture information and transform them into knowledge. The colour of tags, the position of tags on the string, the dynamic position of tag labelled with "1" on the string, recall actions and representations performed in previously activities. Once again, this allows linking different representations and actions to the same mathematical meaning, by supporting long-term memory. According to the principles of the UDL, this contributes to making mathematical meanings accessible to the students.

Discussion and concluding remarques

The first principle of UDL (Providing Multiple Means of Representation) and the related guidelines provide useful references in order to choose artefacts and define tasks which allow overcoming obstacles and difficulties in understanding some of the different meanings of fractions (*part of a whole*, *measure* and *operator*). In order to provide different means of representation, we considered different artefacts: the "placemat", the strip of squared paper, the drawing number line and the "string" on the wall for "hanging out" fractions. Each of them allows students to produce different semiotic representations: pieces of paper as fractional units, coloured sections of the strip as fractions or fractional units, points on the number line, coloured tags labelled with fraction hanging on the string. We note that these representations are of different nature: physical, visual or symbolic. Thus, following the Semiotic Mediation Theory, each artefact allows students to produce situated signs (for example, the sign "enlarge" the tags' position on the string), which will be interpreted as mathematical signs by the teacher's mediation (in the case above, the sign is used by the teacher to introduce the idea of density in the set Q) . The aim is to single out mathematical aspects relevant to the activity and make them accessible to all students through representations; this should allow them to use those mathematical aspects in future. To reach this aim, the design of educational activity (presentation of information, choice of the artefact on which students can act, task design in large sense…) is essential to ensure that all learners have access to knowledge. We note that in our educational sequence about fractions, learners construct mathematical meanings (for instance, the meaning of equivalent fractions) acting on different artefacts (A4 sheet, strip of paper, number line and "strip on the wall") and different representations carried out by those artefacts (fractional units of paper, coloured part of the strip, point on the line and coloured tags composing a "caterpillar"). According to the first principle of the UDL, we state that this is not yet sufficient to ensure the construction of the mathematical meaning, that is to say of usable knowledge (for example, the meaning of equivalent fractions). It is necessary that these different representations be put in relation with each other (for example, two equivalent fractions correspond to the same point on the number line or on the "strip" on the wall because they have the same surface extension as a sheet of A4 paper). Moreover, it is necessary that they be always available to learners during the whole teaching sequence. Thus, according to the second principle of the UDL (Providing Multiple Means of Action and Expression), the action performed by students on different artefacts allows to put into relation different representations associated to the same notion (such as, equivalent fraction) or associated to different meaning of fraction (such as "a part of the whole", in the case of placemat, or "fraction as operator", in the case of strip of paper). The third principle of the UDL (Providing multiple means for engagement) supports the engagement of students in the arithmetic activity concerning fractions. It suggests options to challenge all students, appropriately. Designing activity where a "real" situation requiring a solution was the starting point. As a matter of fact, acquiring new information must be received by the student as a necessity to deal with the challenge posed by the activity. Thus, since the narrative aspect is very important in primary school teaching, the work with the "placemat" started with a letter, sent by a pizzaiolo (pizza chef), where he asks students to realize coloured placemats for his restaurant. The need to make placemats according to the pizzaiolo's requests was, for primary school children, a stimulus to take in information and to process them in order to get new knowledge useful for the activity's aim. To create conditions for the students' self regulation and self assessment in the activity we asked students to work in pairs or in small groups and to compare, by the class discussion, the results of work with the other groups.

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